Student number: ____________________

GIRRAWEEN HIGH SCHOOL

2023 | TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-27
- Allow about 2 hour and 45 minutes for this section

Year 12 Trial HSC Examination - Mathematics 2023
Multiple Choice Answer Sheet

Student Number: ______________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1.	A	○	B	○	C	○	D	○
2.	A	○	B	○	C	○	D	○
3.	A	○	B	○	C	○	D	○
4.	A	○	B	○	C	○	D	○
5.	A	○	B	○	C	○	D	○
6.	A	○	B	○	C	○	D	○
7.	A	○	B	○	C	○	D	○
8.	A	○	B	○	C	○	D	○
9.	A	○	B	○	C	○	D	○
10.	A	○	B	○	C	○	D	○

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 10

Question 1

Given $y = 4\cos\left(2x - \frac{\pi}{4}\right)$ the period and amplitude is

(A) amplitude = 2 and period = π

(B) amplitude = 4 and period = π

(C) amplitude = 4 and period = 2π

(D) amplitude = 4 and period = 3π

Question 2

If $y = 3^{4x-5}$, then $\frac{dy}{dx} =$

(A) $4(3^{4x-5})ln3$

(B) $3(3^{4x-5})ln3$

(C) $4(3^{4x-5})ln4$

(D) $4(3^{4x-4})ln3$

Question 3

Consider the geometric series $1 + (6 - \sqrt{a}) + (6 - \sqrt{a})^2 + (6 - \sqrt{a})^3 + \ldots\ldots\ldots$

If the above series is to have a limiting sum, which of the following statements is correct.

(A) $9 < a < 64$ and $a \neq 36$

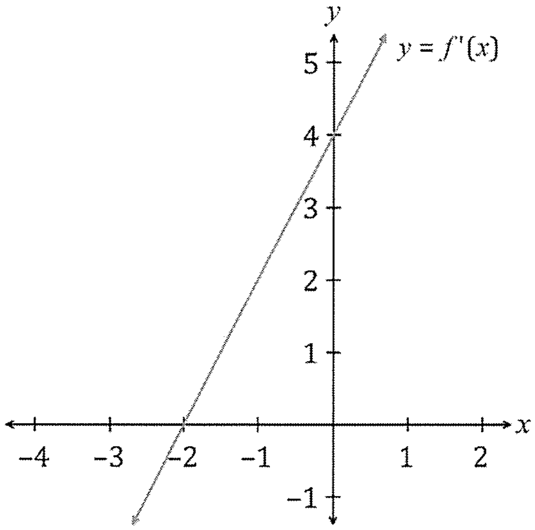
(B) $36 < a < 49$

(C) $25 < a < 49$ and $a \neq 36$

(D) $9 < a < 81$

Question 4

The graph of $y = f'(x)$ is shown below.



Not to scale

The curve $y = f(x)$ has a minimum value of 6.
What is the equation of the curve?

(A) $y = x^2 + 4x + 10$

(B) $y = x^2 - 4x + 10$

(C) $y = x^2 + 4x + 2$

(D) $y = x^2 - 4x + 2$

Question 5

What is the value of $f'(2)$ if $f(x) = \frac{1}{3x}$?

(A) $-\frac{1}{12}$

(B) $-\frac{1}{6}$

(C) $-\frac{3}{4}$

(D) $\frac{1}{13}$

Question 6

Two ordinary dice are rolled. What is the probability that sum of the numbers on the top faces is at least 6?

(A) $\frac{5}{18}$

(B) $\frac{13}{18}$

(C) $\frac{27}{36}$

(D) $\frac{28}{36}$

Question 7

Which of the following represents a function $f(x)$ whose graph has undergone the transformation in the following order?

- Translated left 2 units
- Horizontally dilated by a factor of 3
- Translated down 4 units

(A) $f\left(\frac{x+2}{3}\right) - 4$

(B) $f\big(3(x+2)\big) - 4$

(C) $f\left(\frac{x}{3} + 2\right) - 4$

(D) $f(3x + 2) - 4$

Question 8

Which of the following conditions for $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$ describe the slowing growth of a variable P?

(A) $\frac{dP}{dt} > 0$ and $\frac{d^2P}{dt^2} > 0$

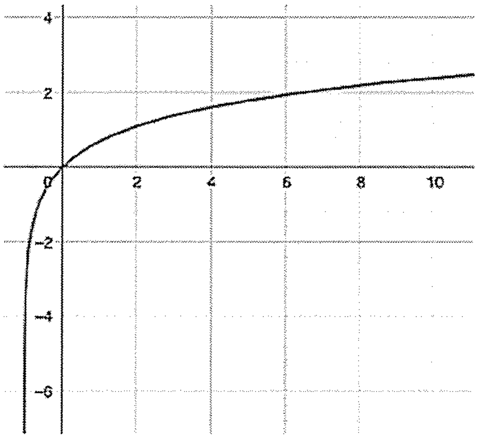
(B) $\frac{dP}{dt} < 0$ and $\frac{d^2P}{dt^2} < 0$

(C) $\frac{dP}{dt} > 0$ and $\frac{d^2P}{dt^2} < 0$

(D) $\frac{dP}{dt} < 0$ and $\frac{d^2P}{dt^2} > 0$

Question 9

The graph below shows the function $y = f(x)$



Which of the following functions best represent $y = f(x)$?

(A) $y = \sqrt{x}$

(B) $y = \ln(x + 1)$

(C) $y = -e^{-x} + 1$

(D) $y = -\frac{1}{x+1} + 1$

Question 10

If $\int_2^5 f(x)dx = 4$, which of the following is $\int_0^3 3f(x + 2)dx$ equal to?

(A) 9

(B) 12

(C) 15

(D) 6

Section II

Answer the questions in the spaces provided.
Your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (4 marks)

a. Factorise fully : $a^2 - bc - b + a^2c$ 2

b. Solve: $\frac{x}{5} - 2 < \frac{x}{2} - 3$ 2

Question 12 (5 marks)

a. The following table represents a probability distribution. The expected value $E(X) = 3.4$.

x	1	2	3	4	5	6
$P(X = x)$	0.1	a	0.3	0.2	0.2	b

Find the value of a and b 2

b. If $x - 3, x$ and $x + 12$ form a geometric sequence, find the value of x and the common ratio. 3

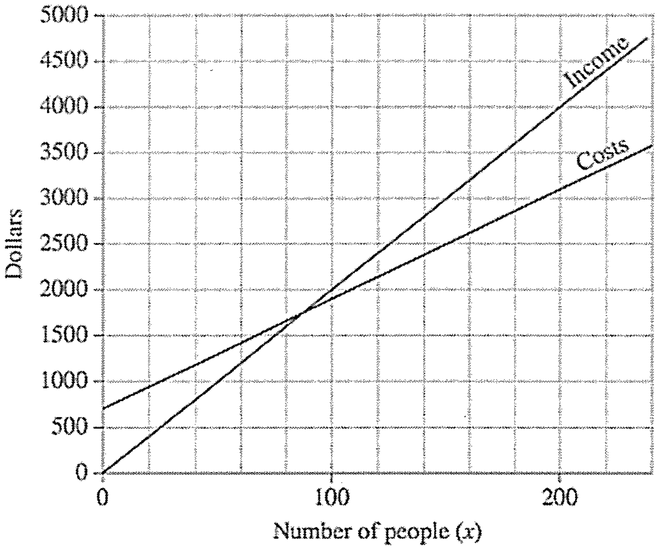
Question 13 (5 marks)

Sue and Mickey are planning a fund-raising dance. They can hire a hall for \$400 and a band for \$300. Refreshments will cost \$12 per person.

i. Write a formula for the cost $\$C$ of running the dance for x people. 1

The graph shows planned income and costs when the ticket price is \$20

Planned income and costs for the dance



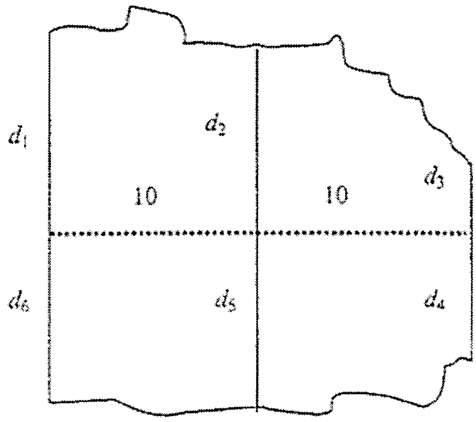
ii. Estimate the minimum number of people needed at the dance to cover the costs. 1

iii. How much profit will be made if 150 people attend the dance? 1

iv. Sue and Mickey want to sell 200 tickets. They want to make a profit of \$1500 . 2
What should be the price of a ticket, assuming all 200 tickets will be sold?

Question 14 (3 marks)

The diagram shows the face of a 20 m wide cliff. The distances $d_1 \ to \ d_6$ are given in the table.



d_1	d_2	d_3	d_4	d_5	d_6
15	14	5.4	8.8	15	14.4

i. Find an estimate for the area of the cliff face using the trapezoidal rule. 2

Give your answer correct to the nearest square metre.

ii. Is the estimate greater than or less than the actual area of the cliff? 1

Justify your answer.

Question 15 (3 marks)

i. Show that $\frac{d}{dx}(x\ln x - x) = \ln x$ 1

ii. Hence Evaluate $\int_1^{e^2} \ln x \; dx$ 2

Leave your answer in exact form.

Question 16 (5 marks)

a. Solve the pair of simultaneous equations 2

$$\log_{10}\frac{x}{y} = 2$$

$$\log_{10} x + \log_{10} y = 4$$

b. Find the equation of the tangent in the simplest gradient-intercept form to the curve

$y = \ln\sqrt{x}$ when $x = e$ 3

Question 17 (7 marks)

a. Find the exact value of x such that $\sec x + 1 = 3$ where $0 \leq x \leq 2\pi$ 2

b. The number of bacteria N a person has after being infected with a virus after t hours is given by:

$$N = 10000e^{0.05t}$$

i. Find the number of bacteria after 10 hours. 1

ii. Find the time required for the number of bacteria to reach 100 000 2

iii. At what rate is the bacteria increasing after 1 day. 2

Question 18 (4 marks)

A single digit from the digits 1 to 9 is written on each of nine cards, so that each digit is used only once.

Huey holds the cards 1 and 2, Dewey holds 3,4 and 5, while Louie holds 6,7,8 and 9.

A card is chosen by randomly choosing one of Huey, Dewey or Louie and then randomly choosing one of that person's cards.

i. Draw a tree diagram to represent the situation. 1

ii. What is the probability that the 9 card is chosen? 1

ii. A two-digit number is to be formed by choosing first the tens digit, and then the units digit. What is the probability that this number is 92? 1

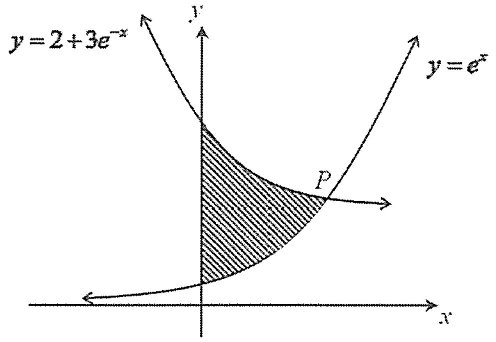
iv. What is the probability that Huey will have no cards left after forming the two-digit number? 1

Question 19 (8 marks)

a. The sum of the first 10 terms of an arithmetic series is 100 and the sum of the next 10 terms is 300. Find the 6th term of the series. 4

b. Sketch the graph of the curve $y = -x^3 + 3x^2 - 1$, labelling the stationary points and point of inflection. DO NOT determine the x intercepts of the curve. 4

Question 20 (6 marks)



The diagram shows the graph of $y = e^x$ and $y = 2 + 3e^{-x}$ intersecting at the point P.

i. Show that x-coordinate of the point P is $\ln 3$. 3

ii. Hence find the exact area of the shaded region. 3

Question 21 (4 marks)

a. Given that the limiting sums S_1 and S_2 of the series both exist, where

$$S_1 = 1 + sin^2x + sin^4x + sin^6x + \cdots$$

$$S_2 = 1 + cos^2x + cos^4x + cos^6x + \cdots$$

i. Show that $S_1 = sec^2x$ and $S_2 = cosec^2x$ 2

ii. Show that $S_1 + S_2 = S_1S_2$ 2

Question 22 (5 marks)

A particle is moving in a straight line. After time t seconds its displacement x metres from a fixed point O on the line is given by $x = t - 3\log_e(t+1)$. The particle returns to its starting point after T seconds.

i. Find when the particle is at rest. 1

ii. Find in simplest exact form the distance travelled by the particle in the first T seconds of its motion. 2

iii. Show that $e^T = (T+1)^3$ 2

Question 23 (6 marks)

A cylindrical container closed at both ends is to be made from thin sheet metal. The container is to have a radius of r cm and a height of h cm such that its volume is

$2000\pi\ cm^3$. $(Volume\ of\ cylinder = \pi r^2 h\ \ and\ Surface\ area\ = 2\pi r^2 + 2\pi rh)$

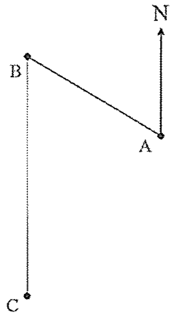
i. Show that the area of sheet metal required to make the container is

$\left(2\pi r^2 + \frac{4000\pi}{r}\right) cm^2$ 2

ii. Hence find the minimum area of sheet metal required to make the container. 4

Question 24 (4 marks)

A plane flies 145 km from point A to point B on a bearing of 305°. The plane then flies 208 km due south to point C before returning to point A.



NOT TO
SCALE

i. Complete the diagram with the information provided and find $\angle ABC$. 2

ii. What distance and on what bearing is the plane's return trip from point C to point A? Round your final answer to the nearest whole number. 2

Question 25 (5 marks)

i. Show that $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$. 2

ii. Hence find $\int (\tan x + \cot x)^2 \, dx$ 3

Question 26 (5 marks)

The rate at which carbon dioxide will be produced by the action of yeast in a dough is given by $\frac{dV}{dt} = \frac{1}{100}(200t - t^2)$ where $V\ cm^3$ is the volume of gas produced after t minutes.

i. At what rate is the gas being produced 2 minutes after the yeast begins to work? 1

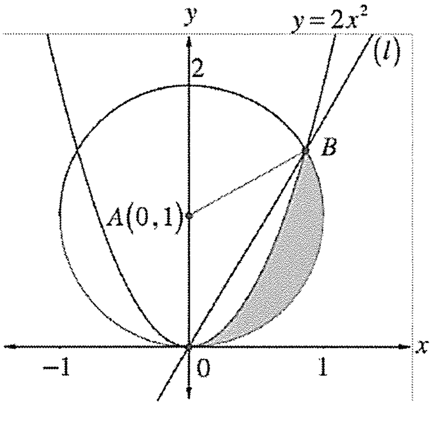
ii. How much carbon dioxide will be produced in the first 5 minutes? 2

iii. Assuming that the given formula is only valid while $\frac{dV}{dt}$ is positive, how long will it be before the reaction stops and how much gas will have been produced altogether? 2

Question 27 (11 marks)

a) Show that $\frac{d}{dx}\{\log_e(1+\sin x)-\log_e\cos x\}=\sec x$ 3

b) The circle centred at $A(0,1)$ and with radius 1 unit intersects the parabola $y=2x^2$ at the origin O and the point B. The line l passes through O and B as shown in the diagram.



i. Show that the coordinates of B are $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$ 2

ii. Find the angle OB makes with the positive x axis 1

iii. Show that $\angle OAB = \frac{2\pi}{3}$ 1

iv. Find the shaded area bounded by the circle and the parabola in the first quadrant as shown in the diagram. 4

END OF EXAMINATION

EXTRA WORKING OUT SPACE

Student number: ____________________

GIRRAWEEN HIGH SCHOOL

2023 | TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

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Total marks: 100

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-27
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Year 12 Trial HSC Examination - Mathematics 2023
Multiple Choice Answer Sheet

Student Number: ______________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1.	A	○	B	●	C	○	D	○
2.	A	●	B	○	C	○	D	○
3.	A	○	B	○	C	●	D	○
4.	A	●	B	○	C	○	D	○
5.	A	●	B	○	C	○	D	○
6.	A	○	B	●	C	○	D	○
7.	A	●	B	○	C	○	D	○
8.	A	○	B	○	C	●	D	○
9.	A	○	B	●	C	○	D	○
10.	A	○	B	●	C	○	D	○

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 10

$y = 4\cos\left[2\left(x - \frac{\pi}{8}\right)\right]$

$y = a\cos bx$

$a = 4$, $T = \frac{2\pi}{b}$

$= \frac{2\pi}{2}$

$= \pi$

Question 1

Given $y = 4\cos\left(2x - \frac{\pi}{4}\right)$ the period and amplitude is

(A) amplitude = 2 and period = π

[✓ (B)] amplitude = 4 and period = π

(C) amplitude = 4 and period = 2π

(D) amplitude = 4 and period = 3π

Question 2

If $y = 3^{4x-5}$, then $\frac{dy}{dx} =$

$y = a^{f(x)}$

$\frac{dy}{dx} = a^{f(x)} \times \ln a \times f'(x)$

$y = 3^{4x-5}$

$\frac{dy}{dx} = 4 \times 3^{4x-5} \times \ln 3$

[✓ (A)] $4(3^{4x-5})ln3$

(B) $3(3^{4x-5})ln3$

(C) $4(3^{4x-5})ln4$

(D) $4(3^{4x-4})ln3$

Question 3

Consider the geometric series $1 + (6 - \sqrt{a}) + (6 - \sqrt{a})^2 + (6 - \sqrt{a})^3 + \ldots\ldots\ldots$

If the above series is to have a limiting sum, which of the following statements is correct.

$r = 6 - \sqrt{a}$

$-1 < 6 - \sqrt{a} < 1$

$-7 < -\sqrt{a} < -5$

$5 < \sqrt{a} < 7$

$25 < a < 49$

(A) $9 < a < 64$ and $a \neq 36$

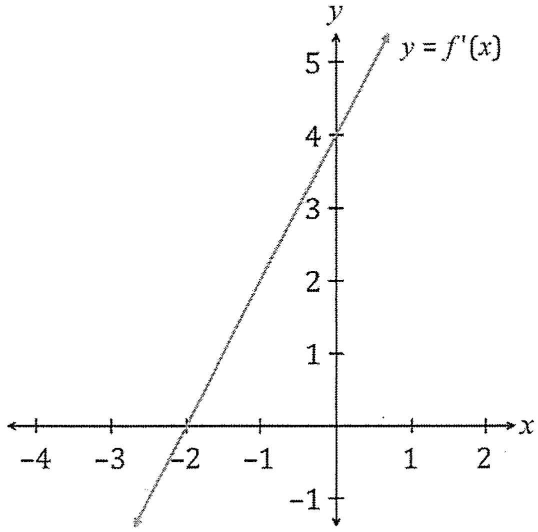
(B) $36 < a < 49$

[✓ (C)] $25 < a < 49$ and $a \neq 36$

(D) $9 < a < 81$

Question 4

The graph of $y = f'(x)$ is shown below.



Not to scale

$f'(x) = 2x+4$ (equation of straight line)

Minima when $f'(x) = 0$ or from graph

$2x+4 = 0$

$x = -2$

$f(x) = \cancel{2}\frac{x^2}{\cancel{2}} + 4x + C$

$f(-2) = (-2)^2 + 4 \times -2 + C =$

$C = 10$

$\therefore f(x) = x^2 + 4x + 10$

The curve $y = f(x)$ has a minimum value of 6.
What is the equation of the curve?

[✓ (A)] $y = x^2 + 4x + 10$

(B) $y = x^2 - 4x + 10$

(C) $y = x^2 + 4x + 2$

(D) $y = x^2 - 4x + 2$

Question 5

What is the value of $f'(2)$ if $f(x) = \frac{1}{3x}$?

$f(x) = \frac{1}{3}x^{-1}$

$f'(x) = -\frac{1}{3}x^{-2} = -\frac{1}{3x}$

$f'(2) = -\frac{1}{3 \times 2^2} = -\frac{1}{12}$

[✓ (A)] $-\frac{1}{12}$

(B) $-\frac{1}{6}$

(C) $-\frac{3}{4}$

(D) $\frac{1}{13}$

Question 6

Two ordinary dice are rolled. What is the probability that sum of the numbers on the top faces is at least 6?

Die I

Die 2

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(A) $\frac{5}{18}$

[✓ (B)] $\frac{13}{18}$

(C) $\frac{27}{36}$

(D) $\frac{28}{36}$

Question 7

Which of the following represents a function $f(x)$ whose graph has undergone the transformation in the following order?

- Translated left 2 units
- Horizontally dilated by a factor of 3
- Translated down 4 units

[✓ (A)] $f\left(\frac{x+2}{3}\right) - 4$

(B) $f(3(x+2)) - 4$

(C) $f\left(\frac{x}{3} + 2\right) - 4$

(D) $f(3x+2) - 4$

Question 8

Which of the following conditions for $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$ describe the slowing growth of a variable P?

(A) $\frac{dP}{dt} > 0$ and $\frac{d^2P}{dt^2} > 0$

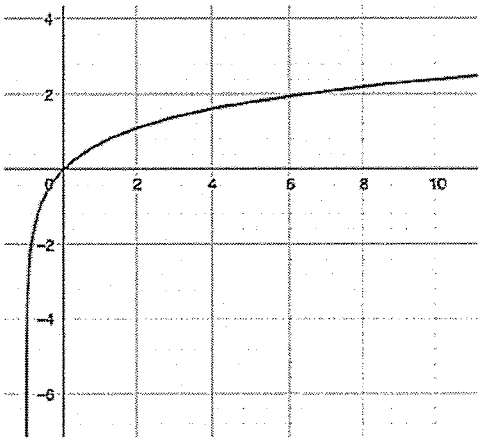
(B) $\frac{dP}{dt} < 0$ and $\frac{d^2P}{dt^2} < 0$

[✓ (C)] $\frac{dP}{dt} > 0$ and $\frac{d^2P}{dt^2} < 0$

(D) $\frac{dP}{dt} < 0$ and $\frac{d^2P}{dt^2} > 0$

Question 9

The graph below shows the function $y = f(x)$



Which of the following functions best represent $y = f(x)$?

(A) $y = \sqrt{x}$

[✓ (B)] $y = \ln(x + 1)$

(C) $y = -e^{-x} + 1$

(D) $y = -\frac{1}{x+1} + 1$

Question 10

If $\int_2^5 f(x)dx = 4$, which of the following is $\int_0^3 3f(x + 2)dx$ equal to?

(A) 9

[✓ (B)] 12

(C) 15

(D) 6

$f(x+2)$ is horizontal translation by 2 units to the left of $f(x)$

$\therefore \int_2^5 f(x)\,dx = \int_0^3 f(x+2)$

$3\int_2^5 f(x)\,dx = \int_0^3 3\,f(x+$

$= 3 \times 4$

$= 12$

Section II

Answer the questions in the spaces provided.
Your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (4 marks)

a. Factorise fully : $a^2 - bc - b + a^2c$ 2

$a^2 + a^2c - bc - b$

$a^2(1+c) - b(c+1)$

$(1+c)(a^2 - b)$

b. Solve: $\frac{x}{5} - 2 < \frac{x}{2} - 3$ 2

$\frac{x}{5} - \frac{x}{2} < -3+2$

$\frac{2x-5x}{10} < -1$

$-3x < -10$

$x > \frac{10}{3}$

Question 12 (5 marks)

a. The following table represents a probability distribution. The expected value $E(X) = 3.4$.

x	1	2	3	4	5	6
$P(X = x)$	0.1	a	0.3	0.2	0.2	b

Find the value of a and b 2

$0.1 + a + 0.3 + 0.2 + 0.2 + b = 1$ (prob. add to 1)

$a + b + 0.8 = 1$

$a + b = 0.2$ — ①

$E(x) = 3.4 = 1 \times 0.1 + 2 \times a + 3 \times 0.3 + 4 \times 0.2 + 5 \times 0.2 + 6 \times b$

$2a + 6b = 0.6$

$a + 3b = 0.3$ — ②

Solve ① & ② simultaneously

② − ① $2b = 0.1$

$b = 0.05$

Sub $b = 0.5$ into ②

$a + 3 \times 0.5 = 0.3$

$a = 0.15$

$\therefore a = 0.15,\ b = 0.05$

b. If $x - 3, x$ and $x + 12$ form a geometric sequence, find the value of x and the common ratio. 3

If T_1, T_2, T_3 form a GP then

$$\frac{T_2}{T_1} = \frac{T_3}{T_2}$$

1 mark $\dfrac{x}{x-3} = \dfrac{x+12}{x}$

$x^2 = (x-3)(x+12)$

$\cancel{x^2} = \cancel{x^2} + 12x - 3x - 36$

$0 = 9x - 36$

$x = 4$

$\therefore$ common ratio $= \dfrac{x}{x-3}$

$= \dfrac{4}{4-3}$

$= 4$

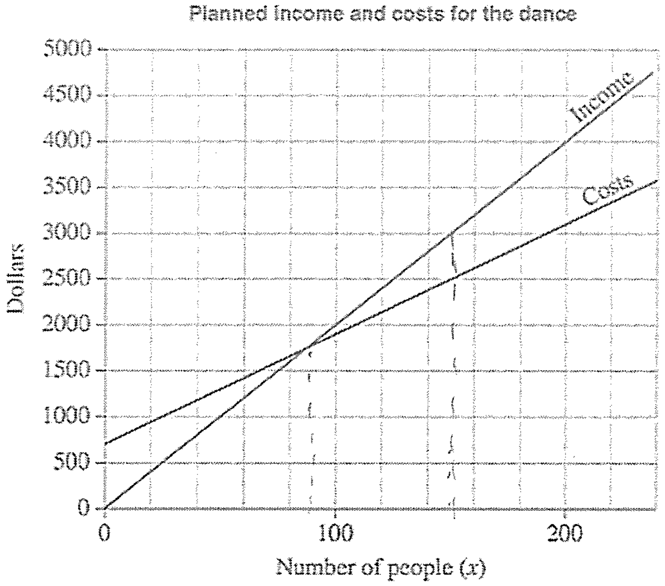
Question 13 (5 marks)

Sue and Mickey are planning a fund-raising dance. They can hire a hall for \$400 and a band for \$300. Refreshments will cost \$12 per person.

i. Write a formula for the cost \$$C$ of running the dance for x people. 1

$C = 12x + 700$

The graph shows planned income and costs when the ticket price is \$20



ii. Estimate the minimum number of people needed at the dance to cover the costs. 1

90 (±1)

iii. How much profit will be made if 150 people attend the dance? 1

Profit $= 150 \times 20 - (700 + 12 \times 150)$

$= 3000 - 2500$

$= \$500$ (or from graph)

iv. Sue and Mickey want to sell 200 tickets. They want to make a profit of \$1500 . 2

What should be the price of a ticket, assuming all 200 tickets will be sold?

Let \$$x$ be the price of each ticket.

$200x - (700 + 12 \times 200) = 1500$

$200x = 4600$

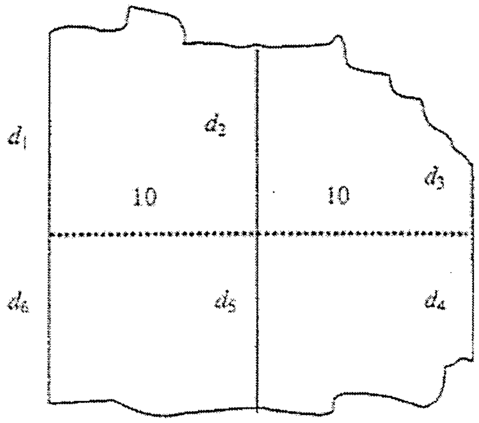
$x = \frac{4600}{200}$

$x = \$23.00$

Price of each ticket \$23

Question 14 (3 marks)

The diagram shows the face of a 20 m wide cliff. The distances $d_1 \ to \ d_6$ are given in the table.



d_1	d_2	d_3	d_4	d_5	d_6
15	14	5.4	8.8	15	14.4

i. Find an estimate for the area of the cliff face using the trapezoidal rule. 2

Give your answer correct to the nearest square metre.

$$A = \frac{10}{2}\left[15 + 5{\cdot}4 + 2\times14\right] + \frac{10}{2}\left[14{\cdot}4 + 15\times2 + 8{\cdot}8\right]$$

$$= 508 \text{ m}^2$$

OR: $A = 5\left[(d_1+d_2) + 2\times(d_2+d_5) + (d_3+d_4)\right]$

$$= 508 \text{ m}^2$$

ii. Is the estimate greater than or less than the actual area of the cliff? 1

Justify your answer.

It is less than the actual area as the curve is concave down.

Question 15 (3 marks)

i. Show that $\frac{d}{dx}(x\ln x - x) = \ln x$ 1

$\frac{d}{dx}\{\ x\ln x - x\ )$

$= \ln x \times 1 + x \times \frac{1}{x} - 1$

$= \ln x + 1 - 1$

$= \ln x$

ii. Hence Evaluate $\int_1^{e^2} \ln x \ dx$ 2

Leave your answer in exact form.

$\because \frac{d}{dx}(x\ln x - x) = \ln x$

$\therefore \int_1^{e^2} \ln x \, dx = \left[x\ln x - x\right]_1^{e^2}$

$= (e^2 \ln e^2 - e^2) - (1 \times \ln 1 - 1)$

$= e^2 \times 2\ln e - e^2 - (0 - 1)$

$= e^2 \times 2 \times 1 - e^2 + 1$

$= e^2 + 1$

Question 16 (5 marks)

a. Solve the pair of simultaneous equations 2

$\log_{10}\frac{x}{y} = 2$ — ①

$\log_{10}x + \log_{10}y = 4$ — ②

from ① $\log_{10}x - \log_{10}y = 2$ — ③ Using log laws

② + ③ $2\log_{10}x = 6$

$\log_{10}x = 3$

$\therefore x = 10^3$

$x = 1000$

② − ③ $2\log_{10}y = 2$

$\log_{10}y = 1$

$y = 10$

$\therefore x = 1000$

$y = 10$

b. Find the equation of the tangent in the simplest gradient-intercept form to the curve

$y = \ln\sqrt{x}$ when $x = e$ 3

$\frac{dy}{dx} = \frac{1}{\sqrt{x}} \times \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2x}$

$\left.\frac{dy}{dx}\right|_{x=e} = \frac{1}{2e}$ 1mark

when $x = e$, $y = \ln\sqrt{e}$

$= \frac{1}{2}\ln e$

$= \frac{1}{2}$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = \frac{1}{2e}(x - e)$

$y - \frac{1}{2} = \frac{x}{2e} - \frac{1}{2}$

$y = \frac{1}{2e}x$

Question 17 (7 marks)

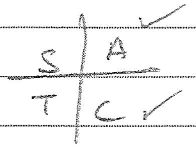
a. Find the exact value of x such that $\sec x + 1 = 3$ where $0 \leq x \leq 2\pi$ 2

$\sec x = 2$

$\frac{1}{\cos x} = 2$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



b. The number of bacteria N a person has after being infected with a virus after t hours is given by:

$$N = 10000e^{0.05t}$$

i. Find the number of bacteria after 10 hours. 1

$N = 10000 \times e^{0 \cdot 05 \times 10}$

$= 16487$

ii. Find the time required for the number of bacteria to reach 100 000 2

$100000 = 10000 \times e^{0 \cdot 05 t}$

$e^{0 \cdot 05 t} = 10$

$0 \cdot 05 t = \ln 10$

$t = \frac{\ln 10}{0 \cdot 05} = 4.6$ hours.

iii. At what rate is the bacteria increasing after 1 day. 2

$\frac{dN}{dt} = 10000 \times 0 \cdot 05 \times e^{0 \cdot 05 t}$

$= 500 \times e^{0 \cdot 05 t}$

$\left.\frac{dN}{dt}\right|_{t=24} = 500 \times e^{0 \cdot 05 \times 24}$

$= 1660$ bac./hour

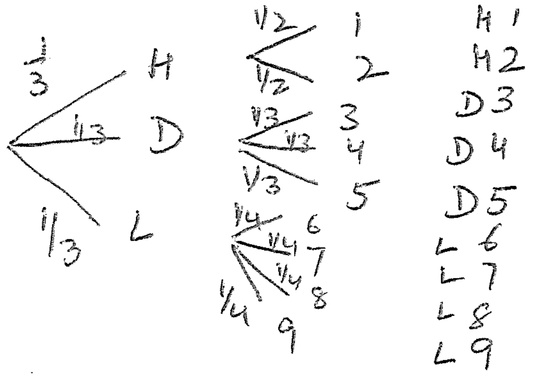
Question 18 (4 marks)

A single digit from the digits 1 to 9 is written on each of nine cards, so that each digit is used only once.

Huey holds the cards 1 and 2, Dewey holds 3,4 and 5, while Louie holds 6,7,8 and 9.

A card is chosen by randomly choosing one of Huey, Dewey or Louie and then randomly choosing one of that person's cards.

i. Draw a tree diagram to represent the situation. 1



ii. What is the probability that the 9 card is chosen? 1

$P(9) = \frac{1}{3} \times \frac{1}{4}$

$= \frac{1}{12}$

ii. A two-digit number is to be formed by choosing first the tens digit, and then the units digit. What is the probability that this number is 92? 1

$\frac{1}{12} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{72}$

iv. What is the probability that Huey will have no cards left after forming the two-digit number? 1

$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

Question 19 (8 marks)

a. The sum of the first 10 terms of an arithmetic series is 100 and the sum of the next 10 terms is 300. Find the 6th term of the series. 4

$S = \frac{n}{2}\left[a + (n-1)d\right]$

$S_{10} = \frac{10}{2}\left[2a + 9d\right]$

$S_{10} = 10a + 45d = 100$

$2a + 9d = 20$ — ①

$S_{20} = 10(2a + 19d) = 100 + 300 = 400$

$2a + 19d = 40$ — ②

Solve ① & ② simultaneously

② − ① $10d = 20$

$d = 2$

sub $d = 2$ into ①

$2a + 9 \times 2 = 20$

$2a = 2$

$a = 1$

$T_n = a + (n-1)d$

$T_6 = 1 + (6-1) \times 2$

$T_6 = 1 + 10$

$= 11$

b. Sketch the graph of the curve $y = -x^3 + 3x^2 - 1$, labelling the stationary points and point of inflection. DO NOT determine the x intercepts of the curve. 4

$y = -x^3 + 3x^2 - 1$ — ①

$y' = -3x^2 + 6x$ — ②

Solve $y' = 0$ for the stationary points

$-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0, \quad x = 2$

$y = -1, \quad y = 3$

To check the nature of the stationary pts.

$y'' = -6x + 6$

$y''(-1) = 6 + 6$

$= 12 > 0$

$\therefore (0,-1)$ minima

$y''(2) = -12 + 6$

$= -6 < 0$

$(2,3)$ maxima

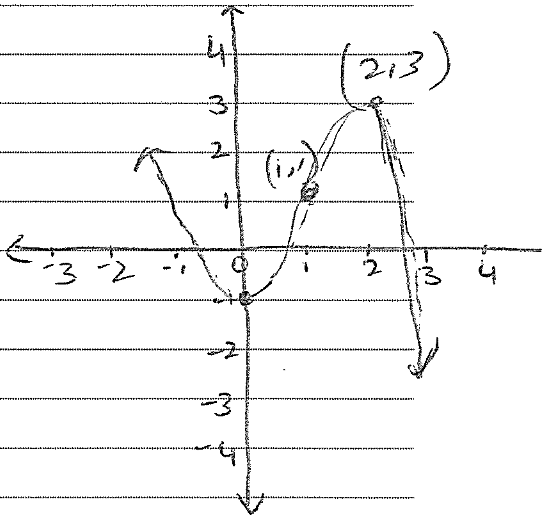
Solve $y'' = 0$ for pts of inflexion

$-6x + 6 = 0$

$x = 1$

$y = -1 + 3 - 1$

$= 1$



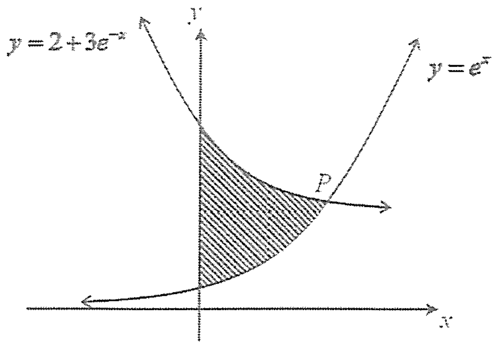
Test for concavity

x	0	1	2
y''	6	0	-6

Concavity changes

$\therefore$ (1,1) is pt. of inflexion

Question 20 (6 marks)



The diagram shows the graph of $y = e^x$ and $y = 2 + 3e^{-x}$ intersecting at the point P.

i. Show that x-coordinate of the point P is $\ln 3$. 3

$y = e^x$ — ① $y = 2 + 3e^{-x}$

Solve ① & ② simultaneously

$2 + 3e^{-x} = e^x$

$2 + \frac{3}{e^x} = e^x$ $\therefore e^x = 3$ or $e^x = -1$

$x = \ln 3$ No solution

$\times e^x$ $2e^x + 3 = e^{2x}$

$e^{2x} - 2e^x - 3 = 0$ $\therefore$ x-co-ordinate of point $P = \ln 3$

Let $e^x = y$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = 3, \quad y = -1$

ii. Hence find the exact area of the shaded region. 3

$A = \int_0^{\ln 3} (2 + 3e^{-x} - e^x)\, dx$

$= \left[2x + \frac{3e^{-x}}{-1} - e^x \right]_0^{\ln 3}$

$= \left[2\ln 3 - 3e^{-\ln 3} - e^{\ln 3} \right] - \left[0 - 3e^0 - e^0 \right]$

$= \left(2\ln 3 - 3 \times \frac{1}{3} - 3 \right) - (-4)$

$= (2\ln 3)\ u^2$

Question 21 (4 marks)

a. Given that the limiting sums S_1 and S_2 of the series both exist, where

$S_1 = 1 + sin^2x + sin^4x + sin^6x + \cdots$

$S_2 = 1 + cos^2x + cos^4x + cos^6x + \cdots$

i. Show that $S_1 = sec^2x$ and $S_2 = cosec^2x$ 2

$$S_1 = \frac{a}{1-r} = \frac{1}{1-\sin^2x} = \frac{1}{\cos^2x} = \sec^2x$$

$$S_2 = \frac{1}{1-\cos^2x} = \frac{1}{\sin^2x} = \csc^2x$$

ii. Show that $S_1 + S_2 = S_1S_2$ 2

$$S_1 + S_2 = \sec^2x + \csc^2x$$

$$= \frac{1}{\cos^2x} + \frac{1}{\sin^2x} \checkmark$$

$$= \frac{\sin^2x + \cos^2x}{\cos^2x\sin^2x} = \frac{1}{\cos^2x\sin^2x}$$

$$= \sec^2x\csc^2x$$

$$S_1 + S_2 = S_1S_2$$

Question 22 (5 marks)

A particle is moving in a straight line. After time t seconds its displacement x metres from a fixed point O on the line is given by $x = t - 3\log_e(t+1)$. The particle returns to its starting point after T seconds.

i. Find when the particle is at rest. 1

$x = t - 3\ln(t+1)$

$\frac{dx}{dt} = 1 - \frac{3}{t+1}$

Particle at rest, $\frac{dx}{dt} = 0$

$1 - \frac{3}{t+1} = 0$

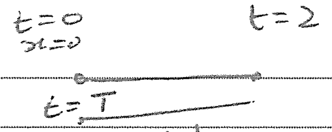
$\frac{3}{t+1} = 1$

$t+1 = 3$

$t = 2$

after 2seconds

ii. Find in simplest exact form the distance travelled by the particle in the first T seconds of its motion. 2



At $t=0$, $x=0$

when $t=2$, $x = |(2-3\ln 3)|$ metres

at $t=T$, particle returns to its ~~original~~ startingpt.

$\therefore$ distance travelled $= 2|(2-3\ln 3)|$ m

$= (6\ln 3 - 4)$ m

iii. Show that $e^T = (T+1)^3$ 2

$x = t - 3\ln(t+1)$

when $t = T$, $x = 0$

$0 = T - 3\ln(T+1)$

$3\ln(T+1) = T$

$T = \ln(T+1)^3$

$(T+1)^3 = e^T$

$e^T = (T+1)^3$

Question 23 (6 marks)

A cylindrical container closed at both ends is to be made from thin sheet metal. The container is to have a radius of r cm and a height of h cm such that its volume is $2000\pi\ cm^3$. $(Volume\ of\ cylinder = \pi r^2 h\ \ and\ Surface\ area\ = 2\pi r^2 + 2\pi rh)$

i. Show that the area of sheet metal required to make the container is $\left(2\pi r^2 + \frac{4000\pi}{r}\right) cm^2$ 2

$$V = \pi r^2 h = 2000\pi$$

$$h = \frac{2000\pi}{\pi r^2} = \frac{2000}{r^2}$$

$$SA = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi \cancel{r} \times \frac{2000}{\cancel{r^2}}$$

$$= \left(2\pi r^2 + \frac{4000}{r}\right) cm^2$$

ii. Hence find the minimum area of sheet metal required to make the container. 4

$$\frac{dA}{dr} = 4\pi r - \frac{4000}{r^2} = 0 \quad \text{for max, min.}$$

$$4\pi r = \frac{4000}{r^2}$$

$$4\pi r^3 = 4000$$

$$r^3 = 1000$$

$$r = 10\ cm.$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{8000}{r^3}$$

$$\left.\frac{d^2A}{dr^2}\right|_{r=10} = 4\pi + \frac{8000\pi}{1000}$$

$$= 12\pi > 0$$

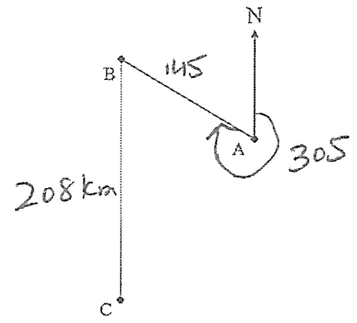
$\therefore$ minima

$$SA = 2\pi \times 10^2 + \frac{4000\pi}{10}$$

$$= (600\pi)\ cm^2$$

Question 24 (4 marks)

A plane flies 145 km from point A to point B on a bearing of 305°. The plane then flies 208 km due south to point C before returning to point A.



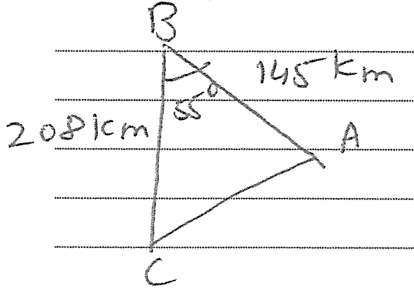
NOT TO SCALE

i. Complete the diagram with the information provided and find $\angle ABC$. 2

$\angle NAB = 360 - 305 = 55^{\circ}$

$\therefore \angle ABC = 55^{\circ}$ (alternate angles on parallel lines)

ii. What distance and on what bearing is the plane's return trip from point C to point A? Round your final answer to the nearest whole number. 2



Using cosine rule

$b^2 = a^2 + c^2 - 2ac \cos B$

$b^2 = 145^2 + 208^2 - 2 \times 145 \times 208 \times \cos 55$

$b = 172.3$ km

$\simeq 172$ km

Using sine rule:

$$\frac{\sin C}{145} = \frac{\sin 55}{172}$$

$$\sin C = \frac{145 \times \sin 55}{172}$$

$C \simeq 44^{\circ}$

Bearing is 044°

Question 25 (5 marks)

i. Show that $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$. 2

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x\,(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

ii. Hence find $\int (\tan x + \cot x)^2\, dx$ 3

$$= \int (\tan^2 x + \cot^2 x + 2)\, dx$$

$$= \int \left[(1 + \tan^2 x) + (1 + \cot^2 x)\right] dx$$

$$= \int \left[\sec^2 x + \operatorname{cosec}^2 x\right] dx$$

$$= \tan x - \cot x + C$$

Question 26 (5 marks)

The rate at which carbon dioxide will be produced by the action of yeast in a dough is given by $\frac{dV}{dt} = \frac{1}{100}(200t - t^2)$ where $V\ cm^3$ is the volume of gas produced after t minutes.

i. At what rate is the gas being produced 2 minutes after the yeast begins to work? 1

$$\left.\frac{dV}{dt}\right|_{t=2} = \frac{1}{100}(200\times2 - 2^2)$$

$$= 3.96\ cm^3/min$$

ii. How much carbon dioxide will be produced in the first 5 minutes? 2

$$V = \frac{1}{100}\int_0^5 (200t - t^2)\,dt$$

$$= \frac{1}{100}\left[100t^2 - \frac{t^3}{3}\right]_0^5 \qquad = \frac{1}{100}\left[2500 - \frac{125}{3}\right]$$

$$= \frac{295}{12} = 24\tfrac{7}{12}\ cm^3$$

iii. Assuming that the given formula is only valid while $\frac{dV}{dt}$ is positive, how long will it be before the reaction stops and how much gas will have been produced altogether? 2

It will stop when $\frac{dV}{dt} = 0$

$$\frac{1}{100}(200t - t^2) = 0$$

$$200t - t^2 = 0$$

$$t(200 - t) = 0$$

$$t = 0, \qquad t = 200$$

$$t > 0, \quad \therefore t = 200 \text{ mins.}$$

$$V = \frac{1}{100}\left[100\times200^2 - \frac{200^3}{3}\right]$$

$$= \frac{40000}{3}\ cm^3$$

Question 27 (11 marks)

a) Show that $\frac{d}{dx}\{\log_e(1+\sin x) - \log_e \cos x\} = \sec x$ 3

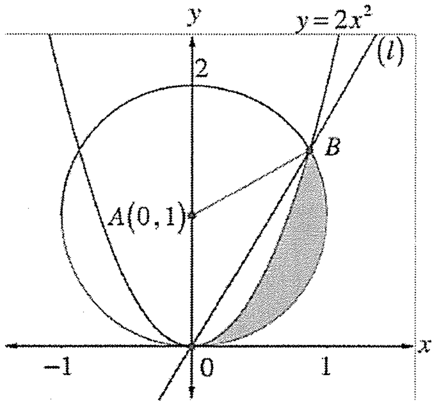
LHS: $\frac{1}{1+\sin x} \times \cos x - \frac{1}{\cos x} \times (-\sin x)$

$= \frac{\cos^2 x + (1+\sin x)\sin x}{(1+\sin x)\cos x}$

$= \frac{\cos^2 x + \sin x + \sin^2 x}{(1+\sin x)\cos x} = \frac{\cos^2 x + \sin^2 x + \sin x}{(1+\sin x)\cos x}$

$= \frac{\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}\cos x} = \sec x = RHS$

b) The circle centred at $A(0,1)$ and with radius 1 unit intersects the parabola $y = 2x^2$ at the origin O and the point B. The line l passes through O and B as shown in the diagram.



i. Show that the coordinates of B are $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ 2

Equation of the circle: $x^2 + (y-1)^2 = 1$ ①

Equation of the Parabola $y = 2x2$ ②

Solve ① & ② simultaneously

$x^2 + (2x^2 - 1)^2 = 1$

$x^2 + 4x^4 - 4x^2 + \cancel{1} = \cancel{1}$

$4x^4 - 3x^2 = 0$

$x^2(4x^2 - 3) = 0$

$x = 0$ or $x = \pm\frac{\sqrt{3}}{2}$

As B is in the 1st quadrant

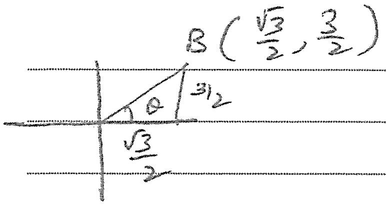
$x = \frac{\sqrt{3}}{2}$

$y = 2x\left(\frac{\sqrt{3}}{2}\right)^2$

$= 3/2$

$\therefore B\left(\frac{\sqrt{3}}{2}, 3/2\right)$

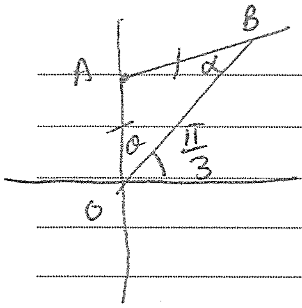
ii. Find the angle OB makes with the positive x axis 1



$$\tan\theta = \frac{3/2}{\sqrt{3}/2} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

iii. Show that $\angle OAB = \frac{2\pi}{3}$ 1



$$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$\alpha = \frac{\pi}{6} \quad (\text{isosceles } \Delta)$$

$$\angle OAB = \pi - \frac{2\pi}{6}$$

$$= \frac{4\pi}{6} = 2\pi/3$$

iv. Find the shaded area bounded by the circle and the parabola in the first quadrant as shown in the diagram. 4

$$\text{Shaded area} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta - \left[\begin{array}{c}\text{area betwn line}\\ \text{and parabola}\end{array}\right]$$

$$\text{Shaded area} = \frac{1}{2}\times 1^2 \times \frac{2\pi}{3} - \frac{1}{2}\times 1^2 \times \sin\frac{2\pi}{3} - \int_0^{\frac{\sqrt{3}}{2}} (\sqrt{3}x - 2x^2)\,dx$$

$$= \frac{\pi}{3} - \frac{1}{2}\times\frac{\sqrt{3}}{2} - \left[\sqrt{3}\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} - \left[\left[\sqrt{3}\times\frac{3}{4}\times\frac{1}{2} - \frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}\sqrt{3}}{\cancel{8}_4}\right] - 0\right]$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} - \left[\frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi}{3} - \cancel{\frac{\sqrt{3}}{4}} - \frac{3\sqrt{3}}{8} + \cancel{\frac{\sqrt{3}}{4}}$$

$$= \left(\frac{\pi}{3} - \frac{3\sqrt{3}}{8}\right) \quad u^2$$

END OF EXAMINATION